Question 1: **(12 marks)**

(a) Express 0.04678 in scientific notation correct to 3 significant figures. **[2]**

(b) Rationalise the denominator of $\frac{3+\sqrt{2}}{2-\sqrt{2}}$. **[3]**

Simplify your answer as far as possible.

(c) Solve for θ if $2\cos\theta = 1$ and $0^\circ \leq \theta \leq 360^\circ$. **[3]**

(d) Solve for k if $3(k-1)-2(k-2)>1$. **[2]**

(e) Express the recurring decimal $0.\dot{2}\dot{2}$ as a fraction in its lowest terms. **[2]**

Question 2: *Start a new sheet of paper* **(12 marks)**

(a) Find the derivatives of each of the following with respect to x:

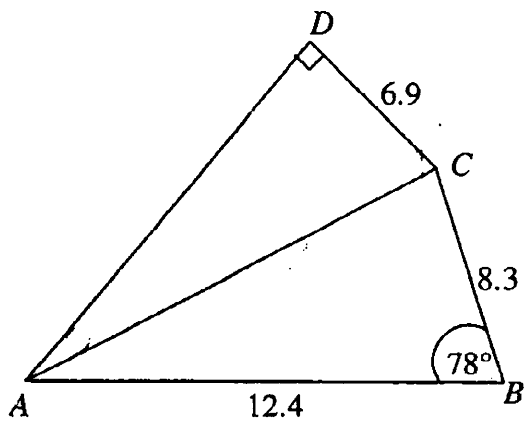
~~(i) $x^3 - 5x^2 + 7x + e^3$~~

(ii) $\ln(x^2 + 5)$

(iii) $\tan(3x)$

(iv) $\dfrac{\sin x}{x^3}$ **[6]**

(b) The quadrilateral $ABCD$ is shown in the diagram – *all measurements are in centimetres.*



(Diagram not to scale)

(i) Find, using the Cosine Rule, or otherwise, the length of AC, correct to one decimal place.

(ii) Find the size of $\angle CAD$, giving your answer to the nearest degree.

(iii) Calculate, correct to one decimal place the area of quadrilateral $ABCD$. **[6]**

Question 3: *Start a new sheet of paper* **(12 marks)**

A circle C, with centre $(-4, 2)$ is tangent to the straight line l with the equation $3x + 4y - 16 = 0$. The perpendicular distance from the centre of this circle to the straight line l is equal to the radius of the circle.

(a) Find the x and y intercepts, A and B, respectively of the line l. [2]

(b) Show that the radius of the circle, C, is 4 units. [3]

(c) Write down the equation of the circle, C, in the form [1]

$$(x - h)^2 + (y - k)^2 = r^2$$

(d) On a suitable number plane, sketch the circle, C and the straight line, l. [2]

(e) Verify that the coordinates of the point of intersection, E, of the circle, C, and the line, l is $(-1.6, 5.2)$. [1]

(f) Find the equation of the line, k, that passes through $(4, -2)$, and is also perpendicular to the line l. [2]

(g) Verify that the circle, C, intersects the y axis at the point D, with coordinates $(0, 2)$. [1]

Question 4: ***Start a new sheet of paper*** **(12 marks)**

(a) (i) Find the number of terms there are in the arithmetic series

$$258 + 251 + 244 + \ldots - 288 - 295 \ ?$$

(ii) Calculate the sum of the arithmetic series in (a)(i) above. **[4]**

(b) (i) Find the discriminant of the quadratic equation

$$x^2 - (2 - k)x + 1 = 0.$$

(ii) Hence, or otherwise, determine the values of k for which the graph of

$$y = x^2 - (2 - k)x + 1$$

cuts the x – axis in two distinct points. **[4]**

(c) A parabola's equation is given by $y = x^2 - 2x - 3$.

(i) Find the focal length.

(ii) Determine the coordinates of the vertex V.

(iii) Find the coordinates of the focus S. **[4]**

Question 5: *Start a new sheet of paper* **(12 marks)**

(a) The common ratio r of a geometric progression satisfies the quadratic equation

$$2r^2 - 3r - 2 = 0.$$

(i) Solve for r

If the sum to infinity of the same progression is 6,

(ii) Explain why, in this case, r can only take on one value. Hence, state the common ratio r.

(iii) Show that the first term a of this progression, is 9. [5]

(b) A population of eels in a certain river has become too large to be supported by its environment and it decreases from time $t = 0$ to a stable level.

The size of the population in thousands of eels is given by $g(t)$, where t is measured in weeks.

The rate of change of the population is given by $g'(t) = -5e^{-2t}$ for $t \geq 0$. It is known that when $t = 0$, $g = 10$.

(i) **Find**, by integrating $g'(t) = -5e^{-2t}$, the size of the population of eels at any time t.

(ii) Sketch the graph of $y = g(t)$ on a number plane.

(iii) State the long-term stability level of the eel population in the river. [7]

Question 6: *Start a new sheet of paper* **(12 marks)**

(a) Consider the function $f(x) = x^3 - x^2 - x + 1$.

(i) By using the *grouping in pairs method for factorising, or otherwise,* factorise fully the expression $x^3 - x^2 - x + 1$.

(ii) Evaluate $f(1)$ **and** $f(-1)$.

(iii) Hence, or otherwise, state the x intercepts of $f(x)$.

(iv) Determine the coordinates **and** the nature of the stationary points on $y = f(x)$.

(v) Find the coordinates of the point(s) of inflexion of $y = f(x)$.

(vi) Hence, sketch the graph of $y = f(x)$ on a number plane, showing all essential features. **[8]**

(b) Forty five percent of a population are of blood group O, 40% are of blood group A and the remainder are neither group O nor group A.

Three people are chosen at random from the population.

By drawing a tree diagram, or otherwise, find the probability that

(i) all three people are of blood group A;

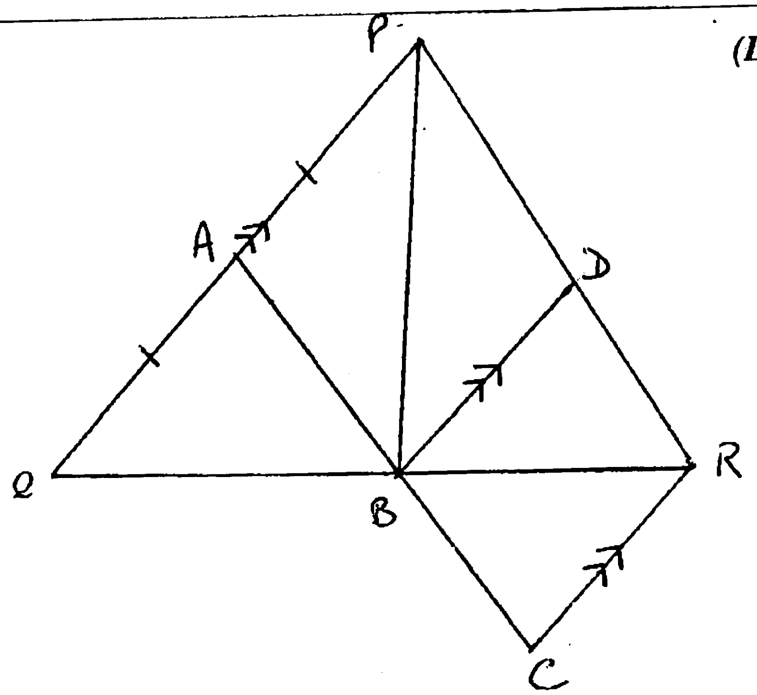
(ii) two of the people are of blood group A and the other is group O;

(iii) there is one person each of group O, group A and neither group O nor group A. **[4]**

Question 8: *Start a new sheet of paper* **(12 marks)**

In triangle PQR, B is a point on QR such that PB is the angle bisector of angle RPQ. A is the midpoint of QP. The straight line joining AB is produced to a point C such that $CR \parallel QP$.

(Diagram is not to scale)



Copy this diagram onto your answer sheet.

(a) Prove, with reasons, that ΔBAQ is similar to ΔBCR. [3]

(b) Hence, state why $\frac{RC}{QA} = \frac{RB}{QB}$. [1]

(c) BD is a line drawn parallel to QP meeting PR at D.

Show that $\frac{RD}{PD} = \frac{RB}{QB}$. [2]

(d) By showing ΔBRD is similar to ΔQRP, or otherwise, show that $\frac{RD}{PD} = \frac{PR}{PQ}$. [3]

(e) Hence, or otherwise, show that $RC = \frac{1}{2}PR$. [3]

Question 7: *Start a new sheet of paper* **(12 marks)**

(a) (i) On a suitable number plane sketch the graph of $y = |x-1|$.

~~(ii) Solve the inequality $|x-1|<1$.~~

(iii) Hence, or otherwise, evaluate $\int_0^2 |x-1|\,dx$. **[4]**

(b) (i) Find $\int \frac{1}{2x-5}\,dx$.

(ii) Find $\int_0^{\frac{\pi}{3}} \cos(6x)\,dx$, leaving your answer in simplified form. **[4]**

(c) Let A be the area enclosed by the curve $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, the x axis and the lines $x = 4$ and $x = 9$.

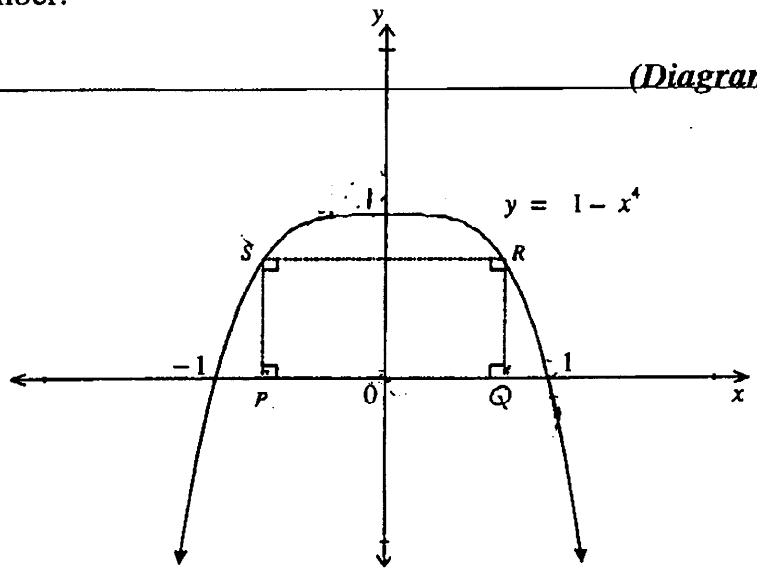
Find the volume of the solid formed by rotating this area through 360° about the x axis.

Leave your answer in simplified form and in terms of π. **[4]**

Question 9: *Start a new sheet of paper* **(12 marks)**

The diagram shows part of the graph of the function $y = 1 - x^4$, where x is any real number.

(Diagram is not to scale)



(a) Calculate I, the area enclosed between the graph of $y = 1 - x^4$ and the x axis. **[3]**

(b) The rectangle $PQRS$ has coordinates

$P(-u, 0)$, $Q(u, 0)$, $R(u, y(\mathrm{u}))$, and $S(-u, y(-\mathrm{u}))$

where $0 < u < 1$.

Let the area of the rectangle $PQRS$ be denoted by $A(u)$.

(i) State why $A(u) = 2u(1 - u^4)$ square units.

(ii) Verify that $A(0.5) = \dfrac{15}{16}$.

(iii) Let u_1 be the value of u for which $A(u)$ is a maximum. **Find** the value of u_1 correct to five significant figures.

(iv) Calculate the maximum value of $A(u)$, giving your answer correct to 5 significant figures.

(v) Show that if $A(u) = uI$, then $u = u_1$. **[9]**

Question 10: *Start a new sheet of paper* **(12 marks)**

Dusty Lifesaver is retiring from the work force. Dusty has a lump sum (from a superannuation fund) of $\$P$ to invest. Dusty is advised that it will take $\$m$ to cover living expenses in any given year after retirement with ~~the remainder invested at r % per annum and compounded annually.~~ The living expense of $\$m$ is taken out of the remainder at the beginning of each year.

Let $R = 1 + \dfrac{r}{100}$ and assuming $P > R(P - m)$ is satisfied.

(a) Show that

(i) after 1 year Dusty has $\$\,R(P - m)$ remaining.

(ii) after 2 years Dusty has $\$\,PR^2 - mR(1 + R)$ remaining.

(iii) after 3 years Dusty has $\$\,PR^3 - mR(1 + R + R^2)$ remaining. **[3]**

(b) Show that $1 + R + R^2 = \dfrac{R^3 - 1}{R - 1}$. **[1]**

(c) By first observing the pattern established in *(a)*, show that after n years the remaining superannuation in dollars is given by

$$PR^n - \frac{mR(R^n - 1)}{R - 1}.$$ **[2]**

(d) By first equating the expression in *(c)* to zero, show that Dusty's superannuation is *exhausted* when

$$R^n = \frac{mR}{P - R(P - m)}.$$ **[3]**

(a) Hence, or otherwise, find the number of years Dusty can survive if $P = \$400\ 000$, $m = \$40\ 000$ and $r = 7\%$ per annum. **[3]**

END OF EXAMINATION

1999 Barker 12 2U Trial MATHS (AES)

Question 1

a) 4.68×10^{-2}

(2)

b) $\dfrac{3+\sqrt{2}}{2-\sqrt{2}} = \dfrac{(3+\sqrt{2})(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})}$

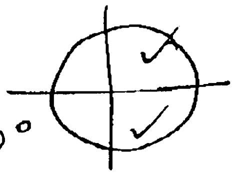
$= \dfrac{6+5\sqrt{2}+2}{4-2}$

$= \dfrac{8+5\sqrt{2}}{2}$

(3)

c) $2\cos\theta = 1 \quad 0° \leq \theta \leq 360°$

$\Rightarrow \cos\theta = \frac{1}{2}$



$\Rightarrow \theta = 60°$ or $300°$

(3)

d) $\Rightarrow 3(k-1) - 2(k-2) > 1$

$\Rightarrow 3k - 3 - 2k + 4 > 1$

$\Rightarrow k + 1 > 1$

$\Rightarrow k > 0$

(2)

e) $0.\dot{2}\dot{2} = \dfrac{22}{99}$

$= \dfrac{2}{9}$

(2)

Question 2

a) i) $3x^2 - 10x + 7$

ii) $\dfrac{2x}{x^2+5}$

iii) $3\sec^2 3x$

iv) $\dfrac{u'v - uv'}{v^2} = \dfrac{\cos x \cdot x^3 - \sin x \cdot 3x^2}{x^6}$

$= \dfrac{x\cos x - 3\sin x}{x^4} \quad x \neq 0$

(6)

b) i) $AC^2 = 12.4^2 + 8.3^2 - 2 \times 12.4 \times 8.3 \times \cos 78°$

$= 179.853$ (3dp) COS RULE

$\Rightarrow AC = 13.4$ correct to 1dp

ii) $\angle CAD = \sin^{-1} \dfrac{6.9}{AC}$

$= 31°$ to nearest degree

iii) Area = Area $\Delta ADC + \Delta ACB$

$= \frac{1}{2} AC \times CD \sin \angle DCA$

$+ \frac{1}{2} AB \times BC \times \sin 78°$

$= \frac{1}{2} \times 13.41 \times 6.9 \times \sin 59°$

$+ \frac{1}{2} \times 12.4 \times 8.3 \times \sin 78°$

$= 36.66 + 50.34$

$= 90.0 \text{ cm}^2$ (1dp)

(6)

Question 3

a) $\ell: 3x+4y-16=0$

x intercept $(5\frac{1}{3}, 0)$

y intercept $(0, 4)$ **(2)**

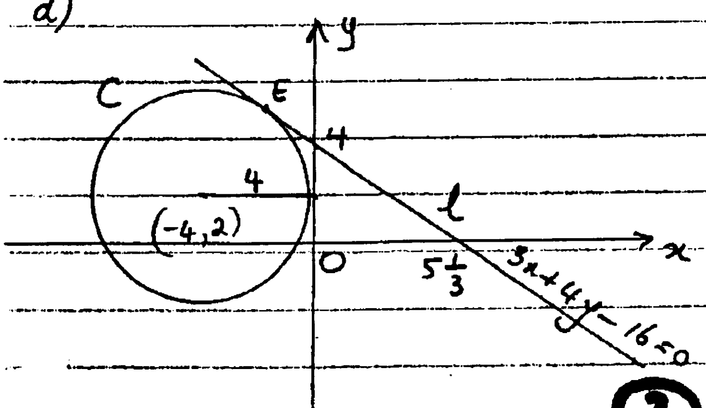
b) $r = d = \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

$= \left|\frac{3(-4)+4(2)-16}{\sqrt{3^2+4^2}}\right|$

$= \frac{20}{5} = 4$ **(3)**

c) $(x+4)^2+(y-2)^2 = 4^2$ **(1)**

d)



(2)

e) E(-1.6, 5.2) sub in

ℓ: $3(-1.6)+4(5.2)-16$

$= -4.8+20.8-16$

$= 0$ ✓ lies on line ℓ.

C: $(-1.6+4)^2+(5.2-2)^2$

$= 2.4^2+3.2^2$

$= 16 = r^2$ ✓ lies on circle

∴ E is on both ∴ E is the intersection. **(1)**

f) $m_\ell = \frac{-3}{4}$

$m_k = \frac{4}{3}$ (4, -2)

$y+2 = \frac{4}{3}(x-4)$

$3y+6 = 4x-16$

$4x-3y-22=0$ **(2)**

g) (0,2) lies on the x axis.

sub (0,2) in C:

$(0+4)^2+(2-2)^2 = 16+0$

$= 16$ ✓ lies on C

∴ (0,2) is intersection D **(1)**

Question 4

a) i) $a = 258$, $d = -7$

$T_n = -295 = a+(n-1)d$

$= 258+(n-1)(-7)$

$= 258-7n+7$

⇒ $7n = 560$

⇒ $n = 80$

ii) $S_n = \frac{n}{2}(a+\ell)$

$= \frac{80}{2}(258-295)$

$= -1480$ **(4)**

b i) $\Delta = b^2-4ac$

$= (-(2-k))^2-4(1)(1)$

$= 4-4k+k^2-4$

$= k(-4+k)$

ii) $\Delta > 0$ for 2 distinct pts

ie $k(k-4) > 0$

i.e. $k < 0$ and $k > 4$ **(4)**

c) i) $y = x^2 - 2x - 3$

$= (x-1)^2 - 1 - 3$

$\Rightarrow y + 4 = (x-1)^2$

$|a| = \frac{1}{4}$

ii) $V = (1, -4)$

iii) $S = (1, -3\frac{3}{4})$

(4)

Question 5

i) $2r^2 - 3r - 2 = 0$

$\Rightarrow (2r+1)(r-2) = 0$

$r = -\frac{1}{2}$ or $r = 2$

If the sequence has a sum to infinity $|r| < 1$ i.e. $r = -\frac{1}{2}$

i) $S_\infty = \frac{a}{1-r} = 6$

$\Rightarrow a = 6 \times (1 - -\frac{1}{2})$

$a = 9$

(5)

b) i) $g'(t) = -5e^{-2t}$

$\Rightarrow g(t) = \frac{5}{2}e^{-2t} + c$

$g(0) = 10$

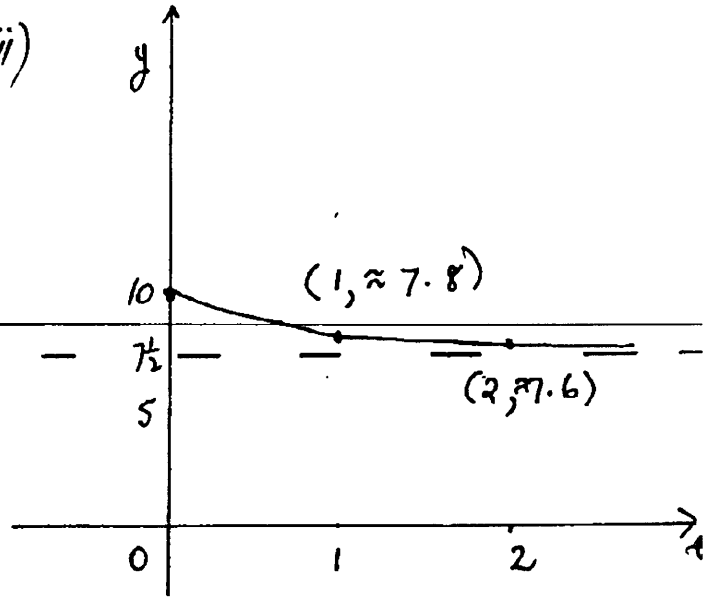
$\Rightarrow 10 = \frac{5}{2}e^{-2\times0} + C$

$\Rightarrow 10 = 2\frac{1}{2} + C$

$\Rightarrow C = 7\frac{1}{2}$

$\Rightarrow g = \frac{5}{2}e^{-2t} + 7\frac{1}{2}$

ii)



iii) 7500 eels

(7)

Question 6

a) i) $x^2(x-1) - 1(x-1)$

$= (x^2-1)(x-1)$

$= (x-1)(x+1)(x-1)$

ii) $f(1) = 0$, $f(-1) = 0$

iii) $(1, 0)$, $(-1, 0)$

iv) For stationary pts $f'(x) = 0$

i.e. $3x^2 - 2x - 1 = 0$

i.e. $(3x+1)(x-1) = 0$

$\Rightarrow x = -\frac{1}{3}$ or $x = 1$

$f''(x) = 6x - 2$

$x = -\frac{1}{3}$ $f''(x) < 0$ max

$x = 1$ $f''(x) > 0$ min.

ctd

MAX at $\left(-\frac{1}{3}, \frac{32}{27}\right)$

MIN at $(1, 0)$

v) Non stationary inflection if

$f''(x) = 0$ (and $f'(x) \neq 0$)

i.e. $6x - 2 = 0$

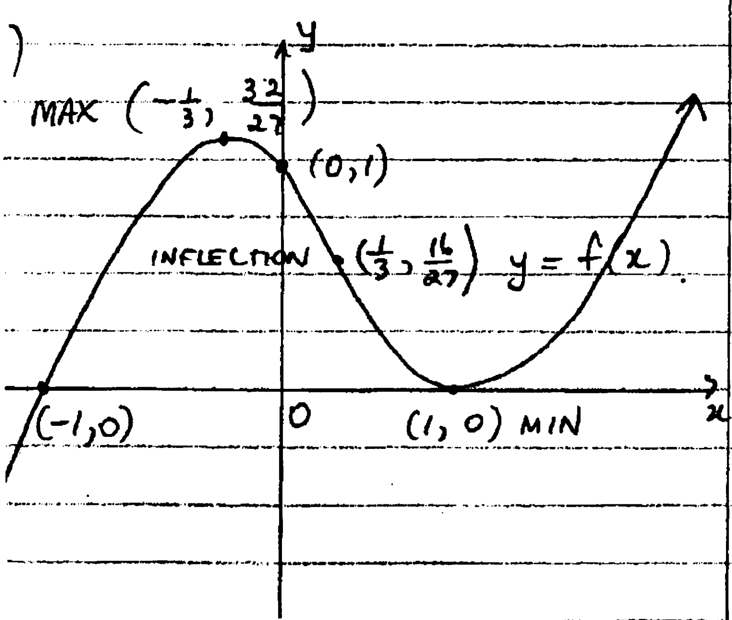
i.e. $x = \frac{1}{3}$

$\Rightarrow y = \frac{16}{27}$

Coords $\left(\frac{1}{3}, \frac{16}{27}\right)$

(8)

)



) $P(O) = 45\%$ $P(A) = 40\%$.

i) $P = (.4)^3 = 0.064$

ii) $P = 3 \times (0.4)^2 \times 0.45$

$= 0.216$

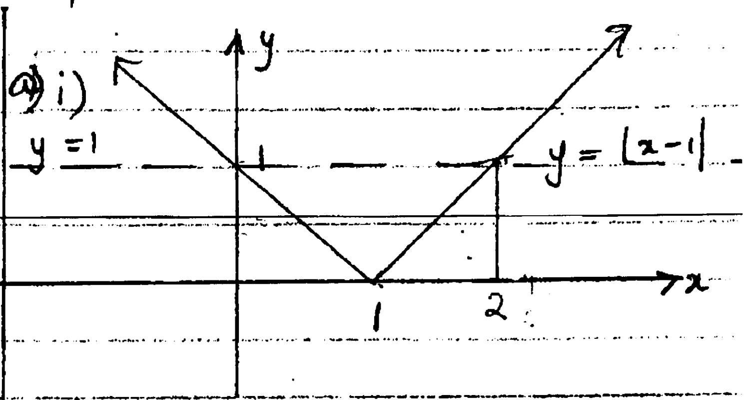
iii) $P = 6 \times 0.4 \times 0.45 \times .15$

$= 0.162$

(4)

Question 7

a) i)



ii) $0 < x < 2$

iii) $\int_0^2 |x-1|\,dx$ = area under $y = |x-1|$ for $0 < x < 2$

$= 2 \times \frac{1}{2} \times 1 \times 1$

$= 1$

(4)

b)

i) $\int \frac{1}{2x-5}\,dx = \frac{1}{2}\ln|2x-5| + C$

ii) $\int_0^{\frac{\pi}{3}} \cos 6x\,dx = \left[\frac{\sin 6x}{6}\right]_0^{\frac{\pi}{3}}$

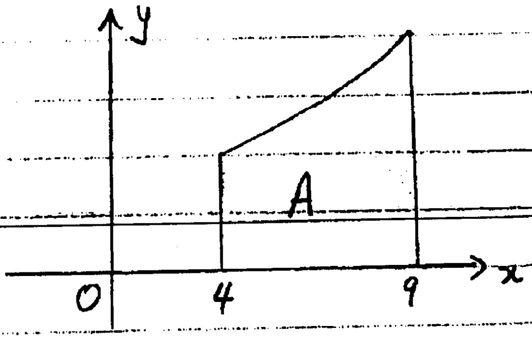
$= \frac{\sin 2\pi}{6} - \frac{\sin 0}{6}$

$= 0$

(4)

7 ctd

c)



$$V = \pi\int_a^b y^2\,dx$$

$$= \pi\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$$

$$= \pi\int_4^9 \left(x + 2 + \frac{1}{x}\right) dx$$

$$= \pi\left(\frac{x^2}{2} + 2x + \ln x\right)_4^9$$

$$= \pi\left(\frac{81}{2} + 18 + \ln 9 - \frac{4^2}{2} - 8 - \ln 4\right)$$

$$= \pi\left(\frac{85}{2} + \ln\frac{9}{4}\right)$$

OR $\pi\left(\frac{85}{2} + 2\ln\frac{3}{2}\right)$ (4)

Question 8

a) In ΔBAQ, ΔBCR

$\angle QAB = \angle RCB$ (alt $\angle$'s $QA \parallel CR$)

$\angle QBA = \angle RBC$ (vert opp $\angle$'s) (3)

$\Rightarrow \Delta BAQ \parallel\!\mid \Delta BCR$ (equiangular)

b) corresponding sides of similar Δ's are in proportion (1)

c) $BD \parallel QP \Rightarrow$

$\frac{RD}{PD} = \frac{RB}{QB}$ (intercept theorem) (2)

d) In ΔBRD, ΔQRP

$\angle R$ is common

$\angle DBR = \angle PQR$ (corresp $\angle$'s $QP \parallel BD$)

$\Rightarrow \Delta BRD \parallel\!\mid \Delta QRP$

$\Rightarrow \frac{RD}{BD} = \frac{RP}{QP}$ * (corresponding sides)

But $\angle DBP = \angle APB$ (alt $\angle$'s $AP \parallel B$...)

and $\angle DPB = \angle APB$ (given)

$\Rightarrow \angle DBP = \angle DPB$

$\Rightarrow \Delta PBD$ is isosceles

$\Rightarrow BD = PD$

$\therefore \frac{RD}{PD} = \frac{RP}{QP}$ from * (3)

e) Proven $\frac{RC}{QA} = \frac{RB}{QB}$, $\frac{RD}{PD} = \frac{RB}{QB}$

$\Rightarrow \frac{RC}{QA} = \frac{RD}{PD}$

but $\frac{RD}{PD} = \frac{RP}{QP}$

$\Rightarrow \frac{RC}{QA} = \frac{RP}{QP}$

$\Rightarrow \frac{RC}{PR} = \frac{QA}{PQ} = \frac{1}{2}$ (A = midpt PQ)

$\Rightarrow RC = \frac{1}{2}PR$ as required. (3)

Question 9

a) $\int_{-1}^{1}(1-x^4)\,dx = \left[x - \frac{x^5}{5}\right]_{-1}^{1}$

$= 2\left(1-\frac{1}{5}\right) = \frac{8}{5}$ **(3)**

b) i) Area $= l \times b$

$A(u) = 2u \times f(u)$

$= 2u \times (1-u^4)$

ii) $A(0.5) = 2(0.5) \times (1-0.5^4)$

$= 1 - \frac{1}{16} = \frac{15}{16}$

iii) $\frac{dA}{du} = \frac{d}{du}(2u - 2u^5)$

$= 2 - 10u^4$

$= 0$ for stationary pt

$\Rightarrow u^4 = \frac{1}{5}$

$\Rightarrow u = \sqrt[4]{\frac{1}{5}}$

$= 0.66874$ to 5 sig figs

Test nature of SP

$\frac{d^2A}{du^2} = -40u$

< 0 for $u = \sqrt[4]{\frac{1}{5}}$

$\therefore$ max S.P. at $u_1 = 0.66874$ (5sf)

iv) $A(u_1) = 2\left(\sqrt[4]{\frac{1}{5}}\right)\left(1-\frac{1}{5}\right)$

$= \frac{8}{5}\left(\sqrt[4]{\frac{1}{5}}\right)$

$= 1.06998$

$= 1.0700$ to 5 sig fig.

v) $A(u) = uI$

$\Rightarrow 2u(1-u^4) = uI$

$= u \times \frac{8}{5}$

$\Rightarrow 1 - u^4 = \frac{4}{5} \quad (0 < u < 1)$

$\Rightarrow u^4 = \frac{1}{5}$

$\Rightarrow u = \sqrt[4]{\frac{1}{5}}$

$= u_1$ (see iii above). **(9)**

Question 10

a) i) After 1 year:

$\$A_1 = (P-m)(1+r\%)$

$= (P-m)R$

$= \$A(P-M)$ remaining

ii) After 2 years

$A_2 = ((P-m)R - m)R$

$= PR^2 - mR^2 + mR$

$= PR^2 - mR(1+R)$

iii) A_3 after 3 yrs

$A_3 = \{[PR^2 - mR(1+R)] - m\}R$

$= PR^3 - mR^2(1+R) - mR$

$= PR^3 - mR(R^2+1) - mR$

$= PR^3 - mR(1+R+R^2)$ **(3)**

b) $(1+R+R^2)(R-1)$

$= R - 1 + R^2 - R + R^3 - R^2$

$= R^3 - 1$

$\Rightarrow 1+R+R^2 = \frac{R^3-1}{R-1}$ **(1)**

c) Following the pattern in a)

$A_n = PR^n - mR(1+R+R^2+\ldots+R^{n-1})$

$= PR^n - mR\frac{(R^n-1)}{R-1}$

(2)

d) If $A_n = 0$

$$PR^n = \frac{mR(R^n - 1)}{R-1}$$

making R^n the subject:

$\Rightarrow R^n(PR - P) = mR\,R^n - mR$

$\Rightarrow R^n(PR - P - mR) = -mR$

$$\Rightarrow R^n = \frac{mR}{P - RP + Rm}$$

$$= \frac{mR}{P - R(P-m)}.$$ (3)

e). $\$P = \$400\,000$ $\$m = \$40000$

$r = 7\%$

$$\Rightarrow 1.07^n = \frac{40000\ (1.07)}{400000 - 1.07(360000)}$$

$$\Rightarrow 1.07^n = \frac{42800}{14800}$$

$$= 2.89189\ldots$$

$$\Rightarrow n = \frac{\log(2.89189\ldots)}{\log 1.07}$$

$= 15.7$ yrs (3 sig fig).

(3)